Security Classification

AD-740 993

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) School of Engineering University of California Irvine, CA 92664	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

MINIMUM SENSITIVE LINEAR FEEDBACK COMPENSATORS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

Allen R. Stubberud, Robert N. Crane

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
21 April 72	5	3

8a. CONTRACT OR GRANT NO. AFOSR 71-2116	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 9769	
c. 61102F	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFOSR - TR - 72 - 0956
d. 681304	

10. DISTRIBUTION STATEMENT

A. Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
TECH, OTHER	Air Force Office of Scientific Research (NM) 1400 Wilson Blvd. Arlington, VA 22209

13. ABSTRACT

In the design of optimal control systems, emphasis is placed on the accuracy of the system mathematical model. If certain modeling parameters deviate from their assumed nominal values, the optimal control may not produce the desired output. A complete theory is developed in this paper for the practical design of linear, nominally equivalent feedback compensators which minimize the output sensitivity to system parameter variations. An example is presented to compare these compensators with those which regulate and stabilize.

6

DD FORM 1 NOV 65 1473

Security Classification

AFOSR - TR - 72 - 0956

MINIMUM SENSITIVE LINEAR FEEDBACK COMPENSATORS

Robert N. Crane
The Johns Hopkins University
Applied Physics Laboratory
8621 Georgia Avenue
Silver Spring, Md. 20910

Allen R. Stubberud
School of Engineering
University of California
Irvine, California 92664

Abstract

In the design of optimal control systems, emphasis is placed on the accuracy of the system mathematical model. If certain modeling parameters deviate from their assumed nominal values, the optimal control may not produce the desired output. A complete theory is developed in this paper for the practical design of linear, nominally equivalent feedback compensators which minimize the output sensitivity to system parameter variations. An example is presented to compare these compensators with those which regulate and stabilize.

1.0 INTRODUCTION

Most of the work on sensitivity reduction in optimal control systems (reference 1) has been with the inclusion of sensitivity terms in the original cost function. This technique trades off the primary design objectives for sensitivity reduction. To achieve the latter, however, a significant deviation from the design goals is usually required. In addition, the original optimal control cannot be realized when the model parameters are at their nominal values.

In this paper, a complete theory for the practical design of linear feedback compensators which minimize output sensitivity is developed. Feedback is used as a second degree of freedom in the optimal control problem to generate a nominally equivalent control function. This function is determined by minimizing the mean square and final value first order sensitivity with a corresponding limitation on the required feedback effort. Necessary and sufficient conditions are developed from which an explicit noniterative solution is obtained for the linear feedback gain term. A comparison example is presented to show the superior sensitivity reduction characteristics of the minimum sensitive gain function relative to regulating and stabilizing controls.

2.0 MINIMUM SENSITIVE CONTROL

2.1 PROBLEM STATEMENT

The solution of an optimal control problem over the time interval [0,T] can be described by the following differential equation:

$$\dot{\underline{x}}_n = \underline{f}(t,\underline{x}_n,\underline{u}_n,\underline{\eta}_n);\quad \underline{x}_n(0) = \underline{x}_{n1} \qquad (1)$$

where $\underline{x}$, $\underline{u}$ and $\underline{\eta}$ are of dimensions n, r and m respectively and the subscript n represents nominal or design values. The function $\underline{f}(\cdot)$ is assumed to be continuous in t and C^1 WRT $\underline{x}$, $\underline{u}$ and $\underline{\eta}$. The solution of (1) is given by:

$$\underline{x}_n(t,\underline{\eta}_n) = \underline{x}_n(t) \qquad (2)$$

which is the desired optimal trajectory. When the optimal control is implemented in the actual system, variations in modeling parameters $\Delta\underline{\eta} = \underline{\eta}-\underline{\eta}_n$ may result in the output $\underline{x}(t)$ deviating significantly from the desired output $\underline{x}_n(t)$. To reduce the output errors caused by parameter variations, a nominally equivalent feedback control function is defined as follows:

$$\underline{u}(t,x) = u_1(t) + K(t)\underline{x}(t) \qquad (3)$$

where K(t) is an (r x n) matrix of time functions and $u_1(t)$ is determined such that

$$u(t,\underline{x}_n) = \underline{u}_n(t). \qquad (4)$$

The corresponding closed loop system is

AFOSR 71-2116

thus

$$\dot{x} = \underline{f}(t,\underline{x},\underline{u}(t,\underline{x}),\underline{n});\ \underline{x}(0) = \underline{x}_{n1} \quad (5)$$

where $\underline{n}$ is the actual parameter vector. Assuming that initial condition errors are accounted for in $u_n(t)$, the problem is to determine $K(t)$ such that, for small variations $\Delta\underline{n}$ from the nominal parameter $\underline{n}_n$, the actual trajectory $\underline{x}(t)$ remains close to $\underline{x}_n(t)$ over the original optimization interval. It is assumed that $\underline{n}$ is known to within a scalar constant, i.e., $\underline{n} = n_a\hat{\underline{n}}$ where n_a is an unknown magnitude operating through a known direction $\hat{\underline{n}}$. From equations (3) and (5), the first order sensitivity vector $\underline{s}(t)$ relative to n_a is described by

$$\dot{\underline{s}} = A(t)\underline{s} + B(t)K(t)\underline{s} + \underline{g}(t);\ s(o)=\underline{s}_o \quad (6)$$

where A, B and $\underline{g}$ represent the partial derivatives of (5) WRT $\underline{x},\underline{u}(\cdot)$ and $\underline{n}_a$ respectively evaluated along the nominal. The initial value of the sensitivity vector $\underline{s}_o$ will normally be zero since the parameter will usually not affect the initial state $\underline{x}_{n1}$.

The sensitivity cost function is defined as follows. Two measures of output sensitivity are

$$\text{mean square} = (1/2)\int_o^T \underline{s}^T Q\underline{s}dt$$

$$\text{final value} = (1/2)\underline{s}^T(T)D\underline{s}(T)$$

where Q and D are positive semi-definite matrices which are continuous in time. The system error is limited by restricting the amount of feedback $K(t)\underline{x}(t)$ or equivalently $K(t)\underline{s}(t)$. This restriction can be included in the cost by the addition of

$$F_1 = (1/2)\int_o^T[\sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}K_{ij}^2 s_j^2]dt$$

where $R_{ij} > 0$ and continuous in time $\forall i,j$. This restricts each state feedback component of the control. The function F_1 can be combined with the output sensitivity measures to yield the following cost functional

$$J(K) = (1/2)\underline{s}^T(T)D\underline{s}(T) + (1/2)\int_o^T[\underline{s}^T Q\underline{s} + \sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}K_{ij}^2 s_j^2]dt \quad (7)$$

which effectively trades off the cost of feedback for reductions in output sensitivity. The problem is thus to determine K(t) such that (7) is minimized subject to (6).

2.2 NECESSARY AND SUFFICIENT CONDITIONS

Necessary conditions for this problem can be obtained from straightforward application of variational methods given in [2]. The Hamiltonian is defined as follows:

$$H_1(t,\underline{s},K,\underline{p}) = -\frac{1}{2}\underline{s}^T Q\underline{s} - \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{n} R_{ij}K_{ij}^2 s_j^2 + \underline{p}^T[A\underline{s} + BK\underline{s} + \underline{g}]. \quad (8)$$

Using this, the optimal gain components are given by

$$K_{ij} = \frac{1}{R_{ij}s_j}\sum_{\ell=1}^{n} p_\ell B_{\ell i} \quad (9)$$

with canonical equations

$$\dot{\underline{p}} = -A^T\underline{p} + Q\underline{s};\ \underline{p}(T) = -D\underline{s}(T) \quad (10)$$

and

$$\dot{\underline{s}} = A\underline{s} + BVB^T\underline{p} + \underline{g};\ \underline{s}(o) = 0 \quad (11)$$

where the (r x r) matrix V is defined by

$$V_{\ell y} = \begin{cases} \sum_{i=1}^{n} \dfrac{1}{R_{\ell i}} & \ell = y \\ 0 & \ell \neq y \end{cases} \quad (12)$$

The canonical equations are time invariant whenever the sensitivity equations and cost matrices are independent of time. The linearity therefore allows a closed form solution for the gain terms given by (9). Note that since $\underline{s}_o = 0$ any value of K(0) will satisfy the optimality conditions. In practice, however, an initial bound must be determined for K(t). The Legendre condition is obtained from (8) as

$$\partial^2 H_1/\partial K_{ij}^2 = -R_{ij}s_j^2 \leq 0$$

$$\partial^2 H_1/\partial K_{ij}\partial K_{\ell m} = 0 \qquad i,j \neq \ell,m \quad (13)$$

The Weierstrass necessary condition is implied by (13) when the extremal is nonsingular (reference [2]). Sufficient conditions are defined by the following theorem:

<u>Theorem 1</u>: The gain matrix $K(t)$ given by (9), (10) and (11) exists on $(0,T]$ as a minimum of (7) subject to (6) if

$$s_j^2(t) > 0 \quad \forall t \in (0,T] \tag{14}$$

where $s_j(t)$ is the j^{th} component of the solution to (6).

<u>Proof</u>: From (13) and (14), reference [2] indicates that sufficient conditions are satisfied if there are no conjugate points. Since the matrices D, Q and R are positive semi-definite, it can be shown that no conjugate points exist. This completes the proof.

From the above theorem, the existence of the minimum sensitive gain is determined mainly by (14) which is a somewhat strong condition and definitely not satisfied for arbitrary cost parameters D, Q and R in (7) and arbitrary functions $\underline{g}(t)$ in (11). This is, however, the price of achieving linearity of the canonical equations (10) and (11). For a given system, cost function and nominal trajectory, these equations can easily be solved to determine if (14) is satisfied. If not, the nonsingular approximate problem formulated in the next section can be employed to obtain the optimal gain.

3.0 A NONSINGULAR SENSITIVITY PROBLEM

The results of the previous section indicate that singular solutions of the minimum sensitivity problem are the major cause for failure of the existence conditions. The problem will be reformulated in this section such that all extremals are nonsingular. As a consequence, the canonical equations become nonlinear and must be solved be approximation or iterative techniques.

3.1 NECESSARY AND SUFFICIENT CONDITIONS

Examination of (9) and (14) reveals that singularities in the optimal gain are synonymous with singular extremals. The cost function (7) will therefore be modified to include a penalty term for large feedback gains as follows:

$$J(K) = \tfrac{1}{2}\underline{s}^T(T)D\underline{s}(T) + \tfrac{1}{2}\int_0^T[\underline{s}^TQ\underline{s} + G]dt$$

with

$$G = \sum_{i=1}^{r}\sum_{j=1}^{n}(R_{ij}K_{ij}^2 s_j^2 + E_{ij}K_{ij}^2) \tag{15}$$

and $E_{ij} > 0 \ \forall i,j$. The Hamiltonian for the problem of minimizing (15) subject to (6) is

$$H_2(t,\underline{s},K,p) = -\tfrac{1}{2}\,\underline{s}^TQ\underline{s} - G + \underline{p}^T[A\underline{s} + BK\underline{s} + \underline{g}]. \tag{16}$$

Using this, the optimal gain components are given by

$$K_{ij} = \frac{s_j}{R_{ij}s_j^2 + E_{ij}} \cdot \sum_{\ell=1}^{n} p_\ell B_{\ell i} \,. \tag{17}$$

with canonical equations

$$\dot{\underline{p}} = -A^T\underline{p} + Q\underline{s} + \underline{m}(\underline{s},\underline{p});\ \underline{p}(T) = -D\underline{s}(T) \tag{18}$$

and

$$\dot{\underline{s}} = A\underline{s} + BZ(s)B^T\underline{p} + \underline{g};\ \underline{s}(o) = 0 \tag{19}$$

where the components of $\underline{m}$ are defined by

$$m_j = -\sum_{i=1}^{r}\frac{E_{ij}s_j}{[R_{ij}s_j^2 + E_{ij}]^2}\cdot\left[\sum_{\ell=1}^{n} p_\ell B_{\ell i}\right]^2 \tag{20}$$

and the $(r \times r)$ matrix Z is defined by

$$Z_{\ell y} = \begin{cases} \displaystyle\sum_{m=1}^{n}\frac{s_m^2}{R_{\ell m}s_m^2 + E_{\ell m}} & \ell = y \\ 0 & \ell \neq y \end{cases} \tag{21}$$

The canonical equations are thus nonlinear in $\underline{s}$ and $\underline{p}$. The Legendre condition is obtained from (10)

$$\partial^2 H_2/\partial K_{ij}\partial K_{\ell m} = \begin{cases} -(R_{ij}s_j^2 + E_{ij}) < 0 & i=\ell,\ j=m \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The Weierstrass necessary condition is implied by (22) since the extremal is nonsingular.

The existence of the optimal gain can be directly proven using Theorem 5 of [3]. With some manipulation, all required hypotheses can be shown to apply. The most difficult is the determination of the constant C for the system and cost inequalities. This can easily be obtained if the term

$$\bar{g} = \sup_{t \in [0,T]} |\underline{g}(t)|$$

is added to the cost J(K), noting that the minimizing gain will be unaltered. Cesari's Theorem is also applicable to the vector case when $R_{ij} = 0, \forall i,j$. In general $R_{ij} > 0$ for some i,j and then the theorem cannot be applied since the gain and state terms are not functionally separable. It is probable, however, that a slight modification can be made to the theorem to prove existence for the general case.

3.2 SOLUTION TECHNIQUES

Since the canonical equations (18) and (19) are nonlinear, they must be solved either by iteration (gradient) or approximation techniques. If it is assumed that E_{ij} is small $\forall i,j$ and that the sensitivity terms in the cost have sufficient weight such that $\underline{s}(t)$ is small, then (18) and (19) can be approximated by a set of linear equations. These assumptions result in $\underline{m}(\cdot) \approx 0$ over [0,T]. Since $E_{ij} > 0\ \forall i,j$ and $\underline{s}(0) = 0$, (21) indicates that $Z(0,E) = 0$. The sensitivity equation (19) therefore initially runs open loop. As the magnitude of $\underline{s}(t)$ increases, the matrix $Z(\underline{s},E)$ approaches V for small E_{ij}. Equations (18) and (19) will thus be approximated as follows:

$$\dot{\underline{p}} = -A^T\underline{p} + Q\underline{s};\ \underline{p}(T) = -D\underline{s}(T) \qquad (23)$$

$$\begin{aligned} \dot{\underline{s}}_1 &= A\underline{s}_1 + \underline{g} &&, \underline{s}_1(0)=0 &&; 0 \le t \le T_1 \\ \dot{\underline{s}}_2 &= A\underline{s}_2 + BVB^T\underline{p} + \underline{g} &&; \underline{s}_2(T_1) = \underline{s}_1(T_1) &&; T_1 < t \le T \end{aligned} \qquad (24)$$

where $T_1 \in (0,T)$ is a design parameter and

$$\underline{s}(t) = \begin{cases} \underline{s}_1(t) & 0 \le t \le T_1 \\ \underline{s}_2(t) & T_1 < t \le T \end{cases} \qquad (25)$$

Equations (23) - (25) can be explicitly solved as a coupled system. The optimal gain K(t) is then determined by (17).

The relationship between the approximate solution given above and that of the singular problem described in Section 2.0 is as follows. The approximation effectively reduces the time interval of optimization and, in doing so, generates an initial sensitivity vector consistent with $\underline{g}(t)$. The problem resulting from some components of $\underline{s}(t)$ approaching zero on $(T_1,T]$ still remains, although this in part dictates the choice of T_1. When this occurs, the approximation of Z by V on $(T_1,T]$ is no longer valid. The choice of T_1 is further complicated by the fact that the desired output sensitivity may not be attained if T_1 is too large. When this approximation cannot be used, recourse must be made to iteration techniques.

4.0 COMPARISON EXAMPLE

The question examined in this section is how much better does the minimum sensitive (MS) gain perform relative to regulator (RG) and stabilizing (ST) gains? A first order example will be described below.

Let the original design system (nominal) be given by

$$\dot{x} = a_n x + u_n ;\quad x(0) = 10 \qquad (26)$$

where $a_n = 1$ and $u_n(t)$ is determined from

$$\min_u \frac{1}{2} \int_0^1 (x^2 + .2u^2)dt \qquad (27)$$

From Section 2.0, the feedback compensator is given by

$$u(t) = u_n(t) + k(t)[y(t) - x_n(t)] \qquad (28)$$

where x_n is the optimal solution of (26). The actual (real world) system is represented by

$$\dot{y} = 1.2y + u(t) \; ; \; y(0) = 10 \qquad (29)$$

where the parameter was varied 20 percent in the unstable direction. Two measures of system error are

$$\text{Mean Square} = \int_0^1 (y-x_n)^2 dt \qquad (30)$$

$$\text{Final Value} = |y(1)-x_n(1)| \quad .$$

The cost of using feedback is measured by

$$\text{Feedback Cost} = \int_0^1 (u-u_n)^2 dt \quad . \qquad (31)$$

Note that if (29) is run open loop ($k(t)=0$) then $u=u_n$ and no cost penalty is incurred.

The MS compensator is determined as a solution to the following problem

$$\min_K \left[\frac{1}{2} ds^2(1) + \frac{1}{2} \int_0^1 (qs^2 + k^2 s^2) dt\right] \qquad (32)$$

subject to

$$\dot{s} = a_n s + ks + x_n(t) \; ; \; s(0) = 0 \qquad (33)$$

which corresponds to that posed in Section 2.0. The regulator gain can also be obtained from (32) and (33) but with $x_n(t)=0$ and $s(0)\neq 0$. For the first order case, the stabilizing gain is a negative constant.

The comparison curves for the minimum sensitive, regulator and stabilizing gains are shown in Figures 1 and 2. For this example, a suitable goal for error reduction with feedback was taken as 10 percent of the open loop error. To achieve this reduction, the figures indicate that the minimum sensitive gain requires at least 30 percent less feedback effort than the regulator and stabilizing gains.

5.0 CONCLUSIONS

The parameter variation problem in optimal control systems has been solved by using feedback as a second degree of freedom in the optimization problem to minimize output sensitivity. Necessary and sufficient conditions were obtained for the minimizing gain function. In addition, an example was presented which indicated that the minimum sensitive gain offered a significant improvement over regulator and stabilizing gains when parameter variations occurred.

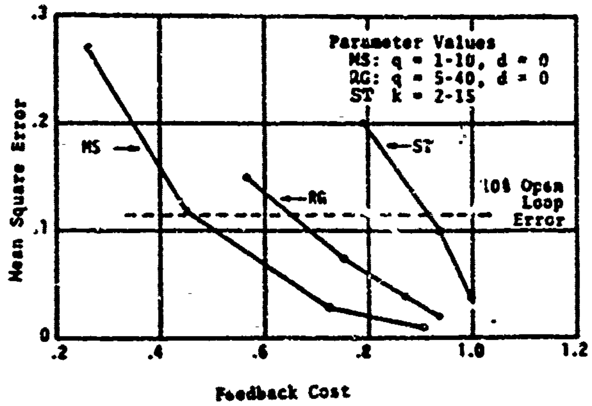

Figure 1. Mean Square Error Comparison Curves

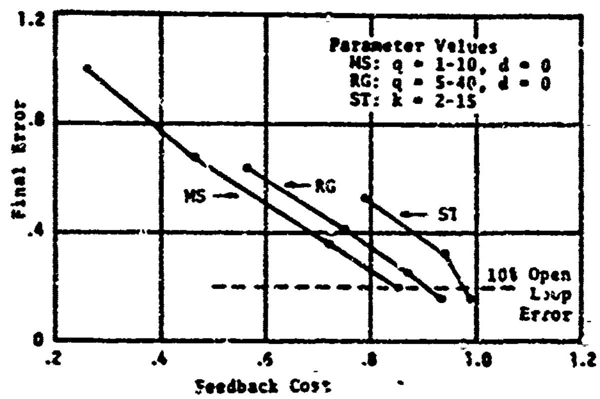

Figure 2. Final Error Comparison Curves

REFERENCES

1) Kreindler, E., "On Minimization of Trajectory Sensitivity," 8:89-97, 1966.

2) Hestenes, M. R., Calculus of Variations and Optimal Control Theory, New York, Wiley, 1966.

3) Cesari, L., "Existence Theorems for Optimal Solutions in Pontryagin and Lagrange Problems, SIAM Journal on Control, 3:475-498, 1966.

ACKNOWLEDGEMENT

This work was partially supported by U.S. Air Force Contracts ~~AFOSR-099-67~~, AFOSR-71-2116 and ~~[illegible]-70-C-0246~~.